Dynamic Feedback Linearization of Second-Order Mechanical Systems

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Abstract

Mechanical systems are most often described by a set of continuous-time, nonlinear, second-order differential equations (SODEs) of a particular structure governed by the covariant derivative. On the other hand, feedback linearization is a well-established technique in control theoretic applications, which allows a nonlinear system to have locally linear properties. Mechanical feedback linearization has also been of recent interest, where one has certain conditions under which feedback linearization simultaneously preserves the mechanical properties of the system.

In this thesis, we visit mechanical feedback linearization and introduce a dynamic compensator to attempt deriving conditions for dynamic mechanical feedback linearization.

Keywords: feedback linearization, mechanical systems, dynamic compensators, differential geometric control, nonlinear control

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List of Symbols

M	manifold
γ	curve/geodesic on M
exp	exponential map on M
TM	tangent bundle of M
$\mathfrak{X}(M)$	space of vector fields on M
X	vector field on M (or TM)
TTM	double tangent bundle of M
φ	diffeomorphism
φ_*	push-forward of the map φ
$T\varphi$	tangent lift of φ
Γ^i_{jk}	Christoffel symbols
g_r	control vector fields
e	uncontrolled vector field
∇	affine connection
$\mathbb{I}$	identity map
$\mathfrak{R}$	Riemannian tensor
ann	annihilator – ann $S = \{f \in V^* : f(v) = 0 \quad \text{for all } v \in S\}$ where $S \subset V$
n	default dimension of M
$\langle \cdot, \cdot \rangle$	inner product
∂_x	partial derivative with respect to x
h	step-size of discretization

Chapter 1

Introduction

1.1 Objective

Feedback linearization has been studied extensively by [1], [3], [4] and has been applied to many real-world problems. Although usually, nonlinear systems in mechanical, aerospace and other engineering domains are second-order mechanical systems which have an underlying mechanical structure and some properties like symmetry, invariance, etc., which must be conserved while performing feedback linearization. This is termed mechanical feedback linearization (or MF-linearization) [6].

A slight modification to the process of feedback linearization by adding a dynamic compensator, leads to technique called dynamic feedback linearization ([2], [5]). This thesis will go over the conditions under which a given mechanical control system is dynamically feedback linearizable while preserving the underlying mechanical structure.

1.2 Preliminaries

Definition 1.2.1 (Control affine systems)**.** *A control affine system* Σ *is a triple* $(M, \mathfrak{g}, f)$ *where:*

- *M is an N-dimensional smooth manifold,*
- *$\mathfrak{g} = (g_1, \ldots, g_m)$ is an m-tuple control vector fields, where $g_i \in \mathfrak{X}(M)$, and*
- *$f \in \mathfrak{X}(M)$ is a drift vector field*

A trajectory of Σ is a piecewise $\mathcal{C}^1$ function of time $z(t) : I \to M$, where I is an interval in $\mathbb{R}$, and that $z(t)$ satisfies the following equation:

$$\Sigma : \dot{z} = f(z) + \sum_{r=1}^{m} g_r(z)u_r, \tag{2.1}$$

where $z \in M$ denotes the state of the system, and $u = (u_1, \ldots, u_m) \in \mathcal{U} \subset \mathbb{R}^m$ are control inputs that belong to a class of admissible controls $\mathcal{U}$ for the system.

Definition 1.2.2 (S-equivalence)**.** *Consider two N-dimensional control-affine systems with the same control space $\mathcal{U} \in \mathbb{R}^m$*

$$\Sigma : \dot{z} = f(z) + \sum_{r=1}^{m} g_r(z)u_r, \quad \tilde{\Sigma} : \dot{\tilde{z}} + \tilde{f}(\tilde{z}) + \sum_{r=1}^{m} \tilde{g}_r(\tilde{z})\tilde{u}_r$$

We can say that Σ *and* $\tilde{\Sigma}$ *are* **state-space equivalence**, *shortly S-equivalent, if there exists a diffeomorphism (or change of coordinates)* $\phi : M \to M$ *such that*

$$\frac{\partial \phi}{\partial z}(z) f(z) = \tilde{f}(\phi(z)), \quad \frac{\partial \phi}{\partial z}(z) g_r(z) = \tilde{g}_r(\phi(z))$$

for all $r = 1, \ldots, m$. *In compact notation, we can also write it as* $\phi_* f = \tilde{f}$ *and* $\phi_* g_r = \tilde{g}_r$.

Definition 1.2.3 (S-linearization)**.** Σ *is state-space linearizable (shortly S-linearizable) if it is S-equivalent to a linear control system* $L\Sigma$. *There exists a local diffeomorphism* $\phi : M \to \mathbb{R}^N$ *such that it simultaneously linearizes the drift vector field* f *and maps the control vector fields* g_r *into constant ones* b_r*:*

$$\frac{\partial \phi}{\partial z}(z) f(z) = A\tilde{z}, \quad \frac{\partial \phi}{\partial z}(z) g_r(z) = b_r$$

and therefore the system in the new coordinates $\tilde{z} = \phi(z)$ *gives:*

$$\dot{\tilde{z}} = \phi_* f + \sum_{r=1}^{m} \phi_* g_r u_r = A\tilde{z} + \sum_{r=1}^{m} b_r u_r \tag{2.2}$$

Definition 1.2.4 (F-equivalence)**.** *Consider two* N*-dimensional control-affine systems*

$$\Sigma : \dot{z} = f(z) + \sum_{r=1}^{m} g_r(z) u_r, \quad \tilde{\Sigma} : \dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \sum_{s=1}^{m} \tilde{g}_s(\tilde{z}) \tilde{u}_s$$

where $z \in M, \tilde{z} \in \tilde{M}, u, \tilde{u} \in \mathbb{R}^m$. *We say that* Σ *and* $\tilde{\Sigma}$ *are* **feedback-equivalent** *or F-equivalent if there exists a diffeomorphism* $\phi : M \to \tilde{M}$ *and an invertible feedback of the form* $u_r = \alpha^r(z) + \sum_{s=1}^{m} \beta_s^r(z) \tilde{u}_s$ *such that*

$$\frac{\partial \phi}{\partial z}(z) \left(f + \sum_{r=1}^{m} g_r \alpha^r \right)(z) = \tilde{f}(\phi(z)), \quad \frac{\partial \phi}{\partial z}(z) \left(\sum_{r=1}^{m} \beta_s^r g_r \right)(z) = \tilde{g}_s(\phi(z))$$

Definition 1.2.5 (F-linearizable)**.** Σ *is* **feedback linearizable** *(F-linearizable) if it is F-equivalent to a linear control system* $L\Sigma$ *of the form* $\dot{\tilde{z}} = A\tilde{z} + B\tilde{u}$. *Thus, there exists a diffeomorphism* $\phi : M \to \mathbb{R}^N$ *and an invertible feedback of the form* $u_r = \alpha^r(z) + \sum_{s=1}^{m} \beta_s^r(z) \tilde{u}_s$ *such that*

$$\dot{\tilde{z}} = \phi_* \left(f + \sum_{r=1}^{m} g_r \alpha^r \right) + \sum_{s=1}^{m} \phi_* \left(\sum_{r=1}^{m} \beta_s^r g_r \right) \tilde{u}_s = A\tilde{z} + \sum_{s=1}^{m} b_s \tilde{u}_s$$

Chapter 2

Mechanical control systems

2.1 Definitions

A mechanical control system $(\mathcal{MS})$ defined on local coordinates $x = (x_1, \dots, x_n)$ on a smooth configuration manifold Q. It takes the form of a second-order differential equation:

$$\ddot{x}_i = -\Gamma^i_{jk}(x)y_j y_k + e_i(x) + \sum_{r=1}^{m} g^i_r(x)u_r, \quad 1 \leqslant i \leqslant n \tag{1.1}$$

Equivalently, we can describe it as two first-order system on the tangent bundle TQ which is the state space of the system using coordinates $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$:

$$\begin{aligned}(\mathcal{MS}): \quad \dot{x}_i &= y_i \\ \dot{y}_i &= -\Gamma^i_{jk}(x)y_j y_k + e_i(x) + \sum_{r=1}^{m} g^i_r(x)u_r, \quad 1 \leqslant i \leqslant n\end{aligned} \tag{1.2}$$

This mechanical system can be written compactly taking $z = (x, y)$ as

$$\dot{z} = F(z) + \sum_{r=1}^{m} G_r(z)u_r \tag{1.3}$$

where $F = y_i \dfrac{\partial}{\partial x_i} + \left(-\Gamma^i_{jk}(x)y_j y_k + e_i(x)\right) \dfrac{\partial}{\partial x_i}$, and $G_r = g^i_r(x)\dfrac{\partial}{\partial y_i}$.

2.2 Mechanical Feedback Equivalence

2.2.1 Mechanical Feedback Transformations

Let MF be a group of transformations generated by:

1. change of coordinates given by diffeomorphisms

$$\begin{aligned}\Phi : &TQ \to T\tilde{Q} \\ &(x, y) \longmapsto (\tilde{x}, \tilde{y}) = (\phi(x), D\phi(x)y)\end{aligned}$$

2. mechanical feedback transformation, denoted by (α, β, γ) of the form

$$u_r = \gamma^r_{jk}(x)y_j y_k + \alpha_r(x) + \sum_{s=1}^{m} \beta^r_s(x)\tilde{u}_s$$

where $\gamma^r_{jk} = \gamma^r_{kj}$

such that the transformed system is linear and mechanical

$$
(L\mathcal{MS}): \quad \begin{aligned} \dot{\tilde{x}}_i &= \tilde{y}_i \\ \dot{\tilde{y}}_i &= E^i_j \tilde{x}_j + \sum_{s=1}^{m} b^i_s \tilde{u}_s \end{aligned} \tag{2.1}
$$

Definition 2.2.1. *Two mechanical control systems $(\mathcal{MS})_{(n,m)} = (Q, \nabla, \mathfrak{g}, e)$ and $(\widetilde{\mathcal{MS}}) = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathfrak{g}}, \tilde{e})$ are mechanical feedback equivalent (or MF-equivalent) if there exists a mechanical transformations $(\phi, \alpha, \beta, \gamma) \in MF$ that maps $(\mathcal{MS})$ to $(\widetilde{\mathcal{MS}})$ according to the following transformations*

$$
\begin{aligned}
\phi : Q \to \tilde{Q} \quad \phi(x) &= \tilde{x} \\
\phi\left(\nabla - \sum_{r=1}^{m} g_r \otimes \gamma^r\right) &= \tilde{\nabla} \\
\phi_*\left(\sum_{r=1}^{m} \beta^r_s g_r\right) &= \tilde{g}_s, \quad 1 \leqslant s \leqslant m \\
\phi_*\left(e + \sum_{r=1}^{m} g_r \alpha^r\right) &= \tilde{e}
\end{aligned}
$$

It is important to note that the group MF preserves the mechanical structure of $(\mathcal{MS})_{(n,m)}$. It can proven such that the Christoffel symbols transform as

$$
\tilde{\Gamma}^i_{ps}(\tilde{x}) = -\frac{\partial^2 \tilde{x}_i}{\partial x_j \partial x_k} \frac{\partial x_j}{\partial \tilde{x}_p} \frac{\partial x_k}{\partial \tilde{x}_s} + \frac{\partial \tilde{x}_i}{\partial x_j} \frac{\partial x_q}{\partial \tilde{x}_p} \frac{\partial x_r}{\partial \tilde{x}_s} \left(\Gamma^i_{qr}(x) - \sum_{l=1}^{m} g^j_l(x) \gamma^l_{qr}(x)\right)
$$

the uncontrolled vector fields is transformed as

$$
\tilde{e}_i(\tilde{x}) = \frac{\partial \tilde{x}_i}{\partial x_j} \left(e_j(x) + \sum_{l=1}^{m} g^j_l(x) \alpha^l(x)\right)
$$

and also control vector fields transform as

$$
\tilde{g}^i_t(\tilde{x}) = \frac{\partial \tilde{x}_i}{\partial x_j} \left(\sum_{l=1}^{m} (x) \beta^l_t(x)\right)
$$

Thus, the mechanical structure is preserved, yielding the transformed dynamics:

$$
\begin{aligned}
\dot{\tilde{x}}_i &= \tilde{y}_i \\
\dot{\tilde{y}}_i &= -\tilde{\Gamma}^i_{ps}(\tilde{x}) \tilde{y}_p \tilde{y}_s + \tilde{e}_i(\tilde{x}) + \sum_{t=1}^{m} \tilde{g}^i_t(\tilde{x}) v_t
\end{aligned}
$$

Chapter 3

Feedback Linearization

3.1 Static Feedback Linearization

Consider a continuous time nonlinear system of the form

$$\Sigma: \quad \dot{x}(t) = f(x(t), u(t)) \tag{1.1}$$

The system Σ is feedback linearizable if there exists a coordinate transformation $\tilde{x} = \varphi(x)$ and a feedback transformation $u = \gamma(x, v)$ such that Σ is a locally a linear system of the form

$$L\Sigma: \quad \dot{\tilde{x}}(t) = A\tilde{x}(t) + Bv(t) \tag{1.2}$$

3.1.1 Example

Consider a motivating example with a system of the form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (1 + 2u(t))x_2(t) \\ u(t) \end{pmatrix} \tag{1.3}$$

Taking $\tilde{x}_1 = x_1 - x_2^2$ and $\tilde{x}_2 = x_2$, we get the diffeomorphism $\varphi(x_1, x_2) = \left(x_1 - x_2^2, x_2\right)$, we get the feedback linearized system:

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} z_2(t) \\ u(t) \end{pmatrix} \tag{1.4}$$

3.2 Dynamic Compensator

Consider a control-affine nonlinear system given by

$$\dot{x} = f(x) + \sum_{r=1}^{m} g_r(x)u_r \tag{2.1}$$

A dynamic compensator is given by introducing a compensator state $w \in \mathbb{R}^q$ such that:

$$\begin{aligned} \dot{w}_p &= \kappa_p(x, w) + \sum_{s=1}^{m} \delta_s^p(x, w)\mu_s \\ u_r &= a_r(x, w) + \sum_{s=1}^{m} b_s^r(x, w)\mu_s \end{aligned} \tag{2.2}$$

Hence, the extended system is given by the dynamics:

$$\begin{aligned}\dot{x} &= f(x) + \sum_{r=1}^{m} g_r(x)\left(a_r(x, w) + \sum_{s=1}^{m} b_s^r(x, w)\mu_s\right)\\ \dot{w}_p &= \kappa_p(x, w) + \sum_{s=1}^{m} \delta_s^p(x, w)\mu_s\end{aligned} \tag{2.3}$$

Writing in compact form, taking $\xi = (x, w)^T$ such that

$$\dot{\xi} = F(\xi) + \sum_{r=1}^{m} G_r(\xi)\mu_r \tag{2.4}$$

3.3 Dynamic Feedback Linearization

Consider the system (2.4) with the dynamic compensator (2.3). If there exists a transformation $z(t) = \Phi(x(t), w(t))$ leading to the linear system:

$$\dot{z}(t) = Az(t) + Bv(t) \tag{3.1}$$

such that $A \in \mathbb{R}^{(n+m)\times(n+m)}$ and $B \in \mathbb{R}^{(n+m)\times q}$, and the linearizing feedback is given by

$$\mu_r = \alpha_r(x, w) + \sum_{s=1}^{m} \beta_s^r(x, w)v_s \tag{3.2}$$

such that

$$\begin{aligned}A\Phi(x) &= DF(\xi)\cdot(F(\xi) + G(\xi)\alpha(\xi))\\ B &= DF(\xi)\beta(\xi)\end{aligned}$$

3.3.1 Example

Consider a motivating example of a system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix}x_2 + 2x_2x_3\\ x_3\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\\ 0\end{pmatrix}u_1 + \begin{pmatrix}2x_2x_4\\ x_4\\ 0\\ 1 + x_3\end{pmatrix}u_2 \tag{3.3}$$

around any $x_0 \in \mathbb{R}^4$ such that $(x_{20}, x_{30}, x_{40}) \neq (0, -1, 0)$ and $u = (u_1, u_2) \in \mathbb{R}^2$. One can easily verify that this system does not satisfy the necessary and sufficient conditions for static feedback linearization given in [4]. However, if we consider a dynamic compensator of the form:

$$\begin{aligned}u_1 &= \mu_1\\ u_2 &= w\\ \dot{w} &= \mu_2\end{aligned}$$

Thus, the extended system dynamics is given by:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{w} \end{pmatrix} = \begin{pmatrix} x_2 + 2x_2(x_3 + x_4 w) \\ x_3 + x_4 w \\ 0 \\ (1 + x_3)w \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mu_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mu_2$$

Thus, we can see that this system is feedback linearizable by the state transformation:

$$z = (z_1, z_2, z_3, z_4, z_5) := \Phi(x_1, x_2, x_3, x_4, w) = \left(x_1 - x_2^2, x_2, x_3 + x_4 w, x_4, (1 + x_3)w\right)$$

and the invertible static feedback

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} := \begin{pmatrix} 1 & x_4 \\ w & 1 + x_3 \end{pmatrix}^{-1} \begin{pmatrix} v_1 - (1 + x_3)w^2 \\ v_2 \end{pmatrix}$$

where $v := (v_1, v_2)$ is the modified control input. This is equivalent to the LTI system:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= v_1 \\ \dot{z}_4 &= z_5 \\ \dot{z}_5 &= v_2 \end{aligned}$$

Hence, the original system is dynamic feedback linearizable around (x_0, u_0).

3.4 Mechanical Feedback Linearization

A mechanical control system $(\mathcal{MS})_{(n,m)} = (Q, \nabla, \mathfrak{g}, e)$ is said to be MF-linearizable if it is MF-equivalent to a linear mechanical system $(L\mathcal{MS})_{(n,m)} = (\mathbb{R}^n, \bar{\nabla}, \mathfrak{b}, E\tilde{x})$, where $\bar{\nabla}$ is the affine connection whose all Christoffel symbols are zero (or flat connection) and $\mathfrak{b} = \{b_1, \dots, b_m\}$ is an m-tuple of constant vector fields. That is, there exists $(\phi, \alpha, \beta, \gamma) \in MF$ such that

$$\begin{aligned} \phi : Q \to \tilde{Q} \quad \phi(x) &= \tilde{x} \\ \phi\left(\nabla - \sum_{r=1}^{m} g_r \otimes \gamma^r\right) &= \bar{\nabla} \\ \phi_*\left(\sum_{r=1}^{m} \beta_s^r g_r\right) &= b_s, \quad 1 \leqslant s \leqslant m \\ \phi_*\left(e + \sum_{r=1}^{m} g_r \alpha^r\right) &= E\tilde{x} \end{aligned}$$

3.5 MF-equivalence of mechanical distributions

The MF-transformations act on the vector fields g_r and e via the triple (ϕ, α, β). It is important to analyse how the distributions are affected by the MF-transformations:

$$\mathcal{E}^0 = \text{span}\{g_r, 1 \leqslant r \leqslant m\}$$
$$\mathcal{E}^j = \text{span}\{ad_e^i g_r, 1 \leqslant r \leqslant m, 0 \leqslant i \leqslant j\}$$

Thus, the transformed distributions are given by:

$$\begin{aligned} \tilde{\mathcal{E}}^0 &= \text{span}\{\tilde{g}_s, 1 \leqslant s \leqslant m\} = \text{span}\left\{\sum_{r=1}^{m} \beta_s^r g_r, 1 \leqslant r \leqslant m\right\} = \mathcal{E}^0 \\ \tilde{\mathcal{E}}^j &= \mathcal{E}^j, \quad 1 \leqslant j \leqslant m \end{aligned} \tag{5.1}$$

It can be proven that this invariance property holds if and only if the distributions $\mathcal{E}^j$ is involutive.

Thus, we state (for proof refer [6]) the sufficient conditions for a mechanical system to be static feedback linearizable while preserving the mechanical structure. The below conditions are thus proven by also taking into consideration that the vertical distribution $\mathcal{V} = \text{span}\left\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right\}$ must be preserved upon transformation.

Theorem 3.5.1. *A mechanical system $(\mathcal{MS})_{(n,m)}$ is said to be mechanical feedback (MF) linearizable, locally around $x_0 \in M$ if and only if, in the neighborhood of x_0, it satisfies the following conditions:*

1. $(ML1)$ *$\mathcal{E}^0$ and $\mathcal{E}^1$ are of constant rank*
2. $(ML2)$ *$\mathcal{E}^0$ is involutive*
3. $(ML3)$ *ann $\mathcal{E}^0 \subset$ ann $\mathfrak{R}$*
4. $(ML4)$ *ann $\mathcal{E}^0 \subset$ ann ∇g_r for all $r : 1 \leqslant r \leqslant m$*
5. $(ML5)$ *ann $\mathcal{E}^1 \subset$ ann $\nabla^2 e$*

For planar mechanical systems, the above conditions are simplified to obtain the conditions stated below:

Proposition 3.5.2. *A planar mechanical system $(\mathcal{MS})_{(2,1)}$ is locally MF-linearizable at $x_0 \in M$ to a controllable $(\mathcal{LMS})_{(2,1)}$, if and only if it satisfies the following conditions:*

1. $(MD1)$ *g and $ad_e g$ are independent*
2. $(MD2)$ *$\nabla_g g \in \mathcal{E}^0$ and $\nabla_{ad_e g} g \in \mathcal{E}^0$*
3. $(MD3)$ *$\nabla^2_{g,ad_e g} ad_e g - \nabla^2_{ad_e g, g} ad_e g \in \mathcal{E}^0$*

3.5.1 Example

Here, we consider an example: a simple mechanical system - the inertia wheel pendulum. The equations of motion are given by

$$\begin{aligned} \mathfrak{m}_{11}\ddot{\theta}^1 + \mathfrak{m}_{12}\ddot{\theta}^2 + c^1 &= 0 \\ \mathfrak{m}_{21}\ddot{\theta}^1 + \mathfrak{m}_{22}\ddot{\theta}^2 &= u \end{aligned} \tag{5.2}$$

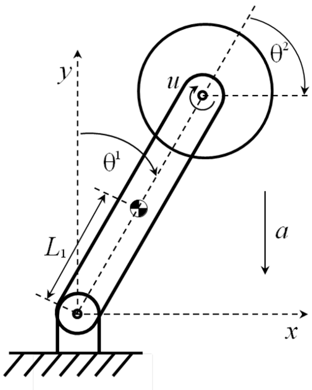

Figure 5.1: The inertia wheel pendulum

where

$$\begin{aligned}\mathfrak{m}_{11} &= m_d + J_2,\ \mathfrak{m}_{12} = \mathfrak{m}_{21} = \mathfrak{m}_{22} = J_2\\ m_d &= L_1^2(m_1+4m_2)+J_1,\ m_0 = aL_1(m_1+2m_2)\\ c^1 &= -m_0\sin\theta^1\end{aligned}$$

Taking $(\theta^1,\theta^2) = (x_1,x_2)$ and correspondingly $(\dot\theta^1,\dot\theta^2) = (y_1,y_2)$, we get the following equations:

$$\begin{aligned}\dot x_1 &= y_1,\ \dot x_2 = y_2\\ \dot y_1 &= e_1 + g_1 u,\ \dot y_2 = e_2 + g_2 u\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}e_1 &= \frac{m_0}{m_d}\sin x_1,\quad g_1 = -\frac{1}{m_d}\\ e_2 &= -\frac{m_0}{m_d}\sin x_1,\quad g_2 = \frac{m_d+J_2}{m_dJ_2}\end{aligned}$$

We will verify $(MD1 - MD3)$ from Proposition 3.5.2, since the mechanical system here is a planar mechanical system.

First, we calculate:

$$\begin{aligned}\mathrm{ad}_e g &= 0 - \begin{pmatrix}\frac{m_0}{m_d}\cos x^1 & 0\\ -\frac{m_0}{m_d}\cos x^1 & 0\end{pmatrix}\begin{pmatrix}-\frac{1}{m_d}\\ \frac{m_d+J_2}{m_dJ_2}\end{pmatrix}\\ &= \begin{pmatrix}\frac{m_0}{m_d^2}\cos x^1\\ -\frac{m_0}{m_d^2}\cos x^1\end{pmatrix}\end{aligned} \tag{5.4}$$

It can be seen that g and $\mathrm{ad}_e g$ are independent (except at $x_1 = \pm\frac{\pi}{2}$). Thus, $MD1$ is satisfied. To verify $MD2$,

$$\begin{aligned}\nabla_g g &= \left(\frac{\partial g_i}{\partial x_j}g_j + \Gamma^i_{jk}g_jg_k\right)\frac{\partial}{\partial x_i} = 0 \in \mathcal{E}^0\\ \nabla_{\mathrm{ad}_e g} g &= 0 \in \mathcal{E}^0\end{aligned} \tag{5.5}$$

which is also verified. Lastly, for $MD3$,

$$\nabla^2_{g,\mathrm{ad}_e g}\mathrm{ad}_e g = \nabla^2_{\mathrm{ad}_e g,g}\mathrm{ad}_e g = \begin{pmatrix} \frac{m_0^2}{m_d^5}\cos^2 x_1 \\ -\frac{m_0^2}{m_d^5}\cos^2 x_1 \end{pmatrix} \tag{5.6}$$

Thus, we have:

$$\nabla^2_{g,\mathrm{ad}_e g}\mathrm{ad}_e g - \nabla^2_{\mathrm{ad}_e g,g}\mathrm{ad}_e g = 0 \in \mathcal{E}^0 \tag{5.7}$$

Therefore, all the conditions $(MD1 - MD3)$ are satisfied, and the given system is MF-Linearizable.

We have the diffeomorphism $\Phi(x, y) = (\phi(x), D\phi(x)y)$, which is given by:

$$\begin{aligned} \tilde{x}_1 &= \frac{m_d + J_2}{J_2}x_1 + x_2, \ \tilde{x}_2 = \frac{m_0}{J_2}\sin x_1 \\ \tilde{y}_1 &= \frac{m_d + J_2}{J_2}y_1 + y_2, \ \tilde{y}_2 = \frac{m_0}{J_2}\cos x_1 y_1 \end{aligned} \tag{5.8}$$

Taking $\tilde{\mathrm{x}} = \begin{pmatrix}\tilde{x}_1 & \tilde{x}_2 & \tilde{y}_1 & \tilde{y}_2\end{pmatrix}^T$, such that the linearized equations become:

$$\frac{d}{dt}\tilde{\mathrm{x}} = A\tilde{\mathrm{x}} + C\tilde{u} \tag{5.9}$$

Here, the matrices $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\tilde{u} = \psi(x, y, u)$ is the auxiliary control, such that:

$$\tilde{u} = -\frac{m_0}{J_2}\sin x_1 y_1^2 + \frac{m_0^2}{2m_d J_2}\sin 2x_1 - \frac{m_0}{m_d J_2}\cos x_1 u \tag{5.10}$$

Chapter 4

Dynamical FL for Mechanical systems

4.1 Mechanical system with dynamic compensator

In this final chapter, we try to derive conditions similar to Theorem 3.5.1, but for a mechanical control system along with a dynamic compensator. The equations of the extended system is given by:

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -\Gamma^i_{jk}(x)y_jy_k + e_i(x) + \sum_{r=1}^{m} g^i_r(x)\left(a_r(x,w) + \sum_{s=1}^{m} b^r_s(x,w)\mu_s\right) \\ \dot{w}_p &= \kappa_p(x,w) + \sum_{s=1}^{m} \delta^p_s(x,w)\mu_s \end{aligned} \tag{1.1}$$

where $w \in \mathbb{R}^q$ is called the compensator state, $\mu \in \mathbb{R}^m$ is the compensator input, and $1 \leqslant i \leqslant n, 1 \leqslant r \leqslant m, 1 \leqslant p \leqslant q$.

Combining these into compact notation with $\xi := (x, y, w)^T$, we have the dynamics:

$$\dot{\xi} = F(\xi) + \sum_{s=1}^{m} G_s(\xi)\mu_s \tag{1.2}$$

where $F = y_i\dfrac{\partial}{\partial x_i} + \left(-\Gamma^i_{jk}y_jy_k + e_i + g^i_r a_r\right)\dfrac{\partial}{\partial y_i} + \kappa_p\dfrac{\partial}{\partial w_p}$ and $G_s = g^i_r b^r_s\dfrac{\partial}{\partial y_i} + \delta^p_s\dfrac{\partial}{\partial w_p}$

Thus, if we have a feedback transformation for this system where, if

$$\mu_r = \alpha_r(\xi) + \gamma^r_{jk}(\xi)y_jy_k + \sum_{s=1}^{m} \beta^r_s(\xi)v_s$$

can lead to a locally linear mechanical system ($L\mathcal{MS}$) given by

$$\dot{z}(t) = Az(t) + Bv(t) \tag{1.3}$$

where $z = \Phi(\xi)$ and v is the linearizing feedback. If such a transformation $(\Phi, \alpha, \beta, \gamma)$ exists, then we have MF-Linearization of the extended system, or in other words, Dynamic MF-Linearization of the original system.

Taking this feedback transformation in the extended system, we obtain

$$\dot{\xi} = F(\xi) + \sum_{s=1}^{m} G_s(\xi) \left(\alpha_s(\xi) + \gamma^s_{jk}(\xi) y_j y_k + \sum_{l=1}^{m} \beta^s_l(\xi) v_l \right)$$

This can again be compactly written as a first-order ODE on $TQ \times \mathbb{R}^q$ given by:

$$\dot{\xi}_j = \tilde{F}_j(\xi) + \sum_{l=1}^{m} \tilde{G}^j_l(\xi) v_l$$

for $j = 1, \ldots, 2n + q$ where,

$$\tilde{F} = y_i \frac{\partial}{\partial x_i} + \left(-(\Gamma^i_{jk} - g^i_r b^r_s \gamma^s_{jk}) y_j y_k + e_i + g^i_r a_r + g^i_r b^r_s \alpha_s \right) \frac{\partial}{\partial y_i} + \left(\kappa_p + \delta^p_s \alpha_s + \delta^p_s \gamma^s_{jk} y_j y_k \right) \frac{\partial}{\partial w_p}$$

and

$$\tilde{G}_l = g^i_r b^r_s \beta^s_l \frac{\partial}{\partial y_i} + \delta^p_s \beta^s_l \frac{\partial}{\partial w_p}$$

We can thus find conditions and corresponding transformations for mechanical feedback equivalence of this extended system which, if MF-equivalent to a linear mechanical system of the form 1.3, where $z \in \mathbb{R}^{2n} \times \mathbb{R}^q$, then we can establish conditions for dynamic MF-Linearization of system 1.2.

Proposition 4.1.1. *The system 1.2 is dynamically MF-linearizarable to a linear mechanical system 1.3 if the following necessary conditions are satisfied:*

1. *The distribution $\tilde{\mathcal{E}}_k = span\{ad_{\tilde{F}} \tilde{G}_s, 1 \leqslant j \leqslant k\}$ must be involutive and constant rank*
2. *The rank $(\tilde{G}_1, \ldots, ad_{\tilde{F}} \tilde{G}_m, ad_{\tilde{F}} \tilde{G}_m, \ldots, ad^{2n+p-1}_{\tilde{F}} \tilde{G}_m) = 2n + p$*
3. *The diffeomorphism must preserve the vertical distribution*

$$\Phi_* \left(span \left\{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_q} \right\} \right) = span \left\{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_q} \right\}$$

4. *The maps (b, γ) must be such that $\Gamma^i_{jk} - g^i_r b^r_s \gamma^s_{jk} = 0$, for all $1 \leqslant i \leqslant n$.*

Note that these are not sufficient conditions. They are derived by following a procedure similar to conditions obtained in Theorem 3.5.1.

Future work involves deriving the sufficient conditions and applying to mechanical control systems like inverted pole and cart system, inverted wheel pendulum, etc. This remains an open problem since there may exist very few mechanical systems belonging to this class of dynamically MF-linearizable systems.

Bibliography

[1] Roger W Brockett. "Feedback invariants for nonlinear systems". In: *IFAC Proceedings Volumes* 11.1 (1978), pp. 1115–1120.

[2] B. Charlet, J. Lévine, and R. Marino. "On dynamic feedback linearization". In: *Systems and Control Letters* 13.2 (1989), pp. 143–151. issn: 0167-6911. doi: https://doi.org/10.1016/0167-6911(89)90031-5. url: https://www.sciencedirect.com/science/article/pii/0167691189900315.

[3] JW Grizzle. "A linear algebraic framework for the analysis of discrete-time nonlinear systems". In: *SIAM Journal on Control and Optimization* 31.4 (1993), pp. 1026–1044.

[4] B Jacubczyk and W Respondek. "On linearization of control systems". In: *Bul. L'acad Pol. Sciense* 28.9-10 (1980), pp. 517–522.

[5] Hong-Gi Lee. "Dynamic Feedback Linearization". In: *Linearization of Nonlinear Control Systems*. Singapore: Springer Nature Singapore, 2022, pp. 227–261.

[6] Marcin Nowicki. *Feedback linearization of mechanical control systems*. General Mathematics, Normandie Université, 2020.